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Quantum Approximate Markov Chains and the Locality of Entanglement Spectrum

Fernando G.S.L. Brandão

Caltech

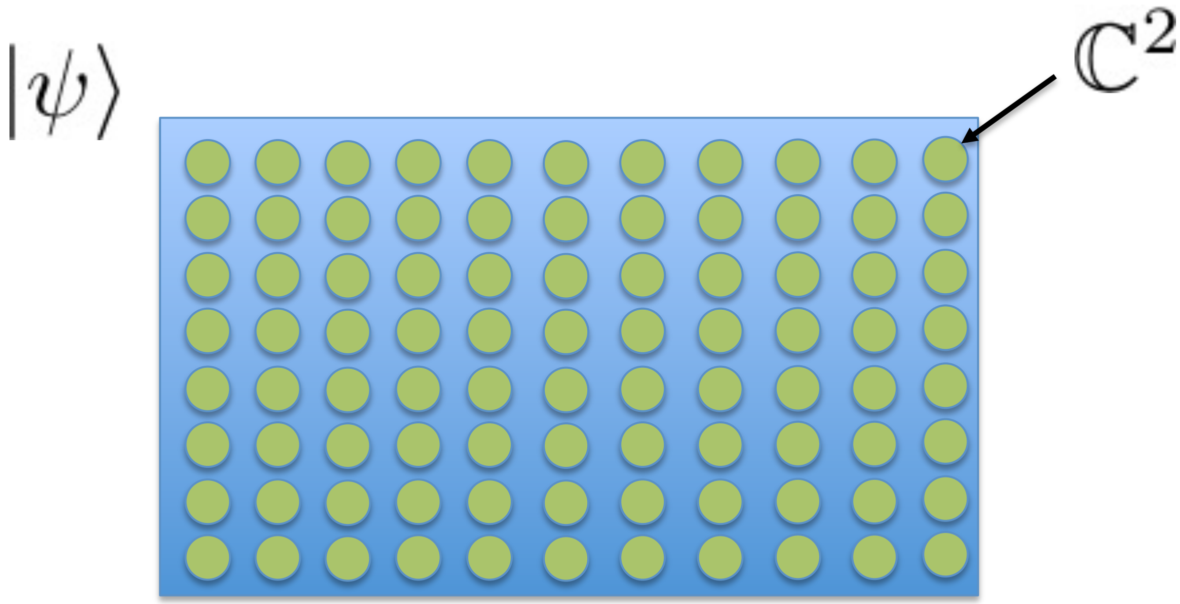
based on joint work with

Kohtaro Kato

University of Tokyo

QMath 2016

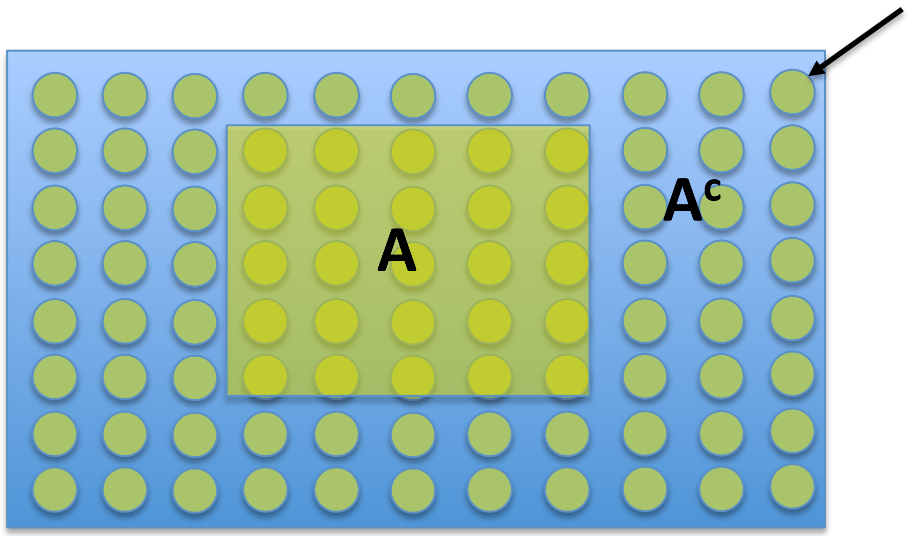
Entanglement in Many-Body Quantum States



Entanglement in Many-Body Quantum States

$|\psi\rangle_{AA^c}$

$\mathbb{C}^2$



Entanglement Entropy: $S(A) = -\mathrm{tr}(\rho_A \log \rho_A)$

Entanglement in Many-Body Quantum States

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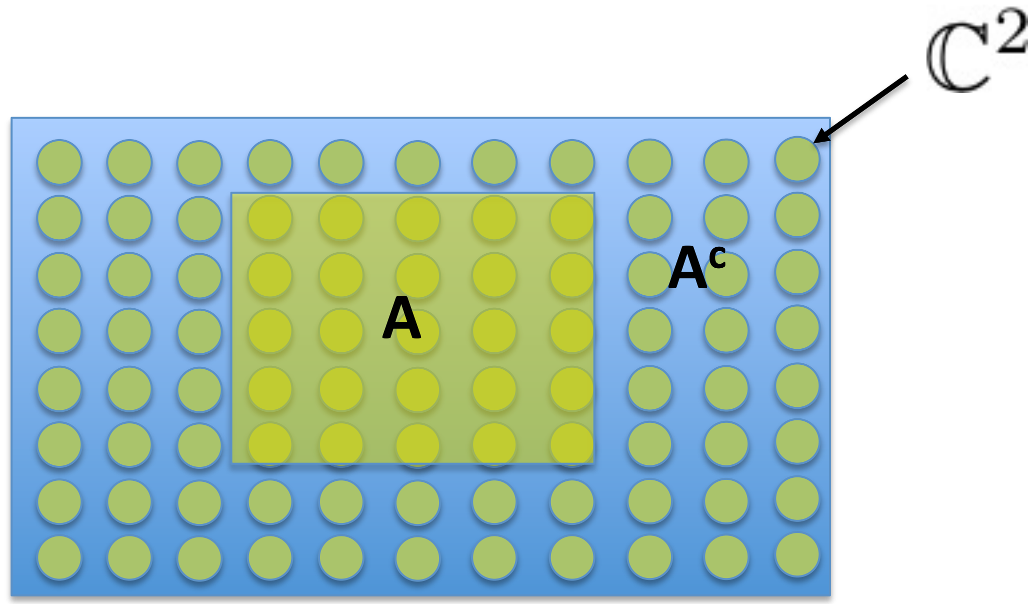

Entanglement Entropy: $S(A) = -\mathrm{tr}(\rho_A \log \rho_A)$

For generic quantum states: $S(X) \approx \mathrm{vol}(X)$ (Page '93)

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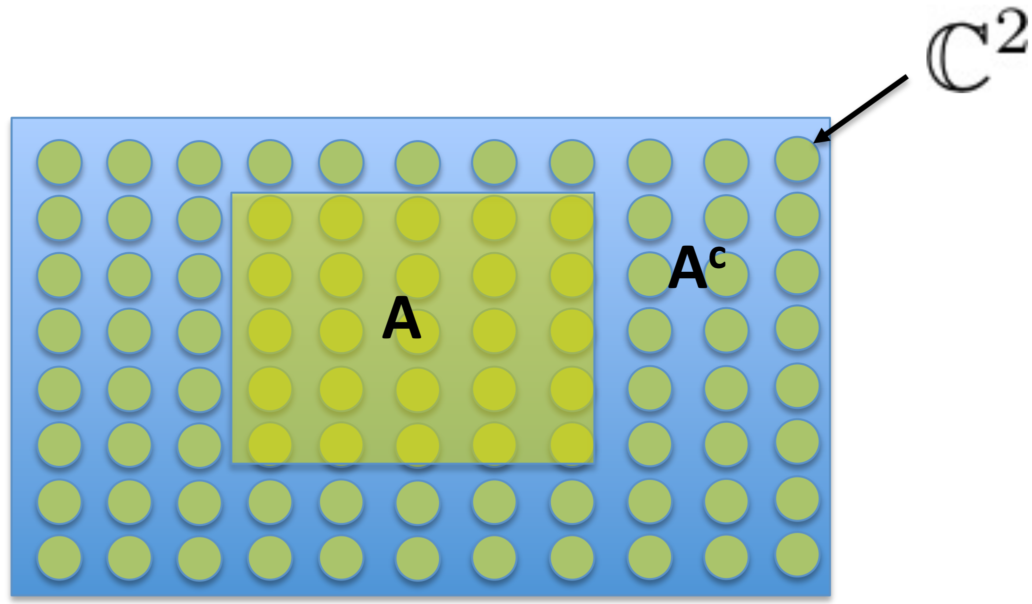
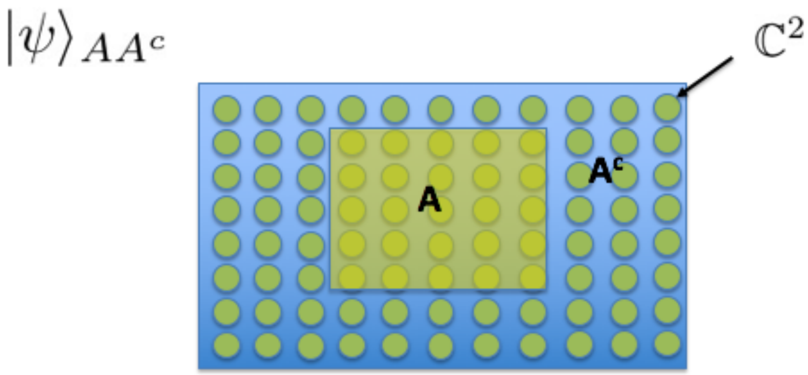


Entanglement Entropy: $S(A) = -\mathrm{tr}(\rho_A \log \rho_A)$

For generic quantum states: $S(X) \approx \mathrm{vol}(X)$ (Page '93)

What's the behavior of EE for interesting states of matter?

Area Law



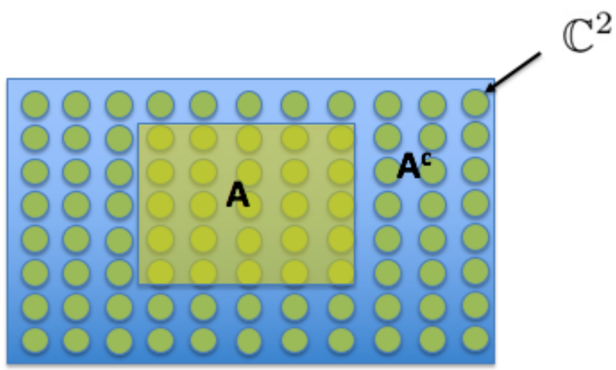
Entanglement is "localized", concentrated around the boundary

For every region *X*: $S(X) = \alpha|\partial X| - \gamma + \ldots$

e.g. gapped models, 2+1 CFT (from RT formula)

Area Law

$|\psi\rangle_{AA^c}$



Entanglement is "localized", concentrated around the boundary

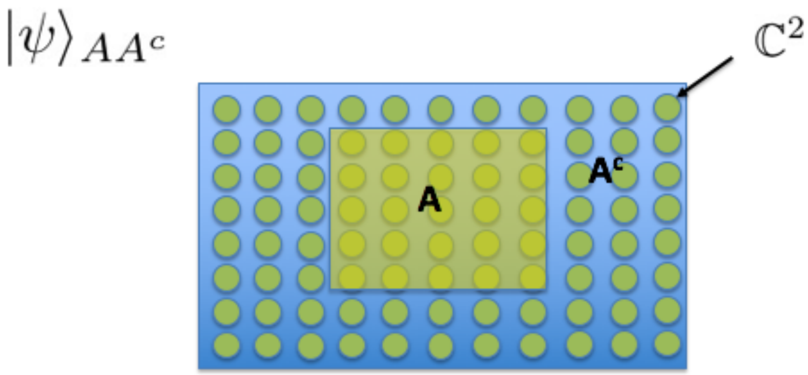
For every region *X*: $S(X) = \alpha|\partial X| - \gamma + \ldots$

ɣ: Topological EE
(signature topological order)

$$\gamma = \log \mathcal{D}, \quad \mathcal{D} = \sqrt{\sum_a d_a^2}$$

D: Quantum dimension

Area Law



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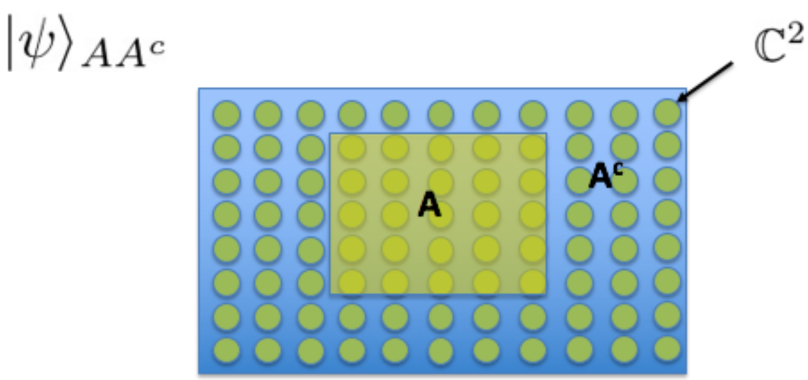
For every region *X*: $S(X) = \alpha|\partial X| - \gamma + \ldots$

- Topological EE quantifies "non-local entanglement"

(Kitaev '12) ɣ = 0 : state is adiabatically connected to trivial phase
(Kim '13) log(N) ≤ 2ɣ   N := number topologically protected states
⋮

Area Law



Entanglement is “localized”, concentrated around the boundary

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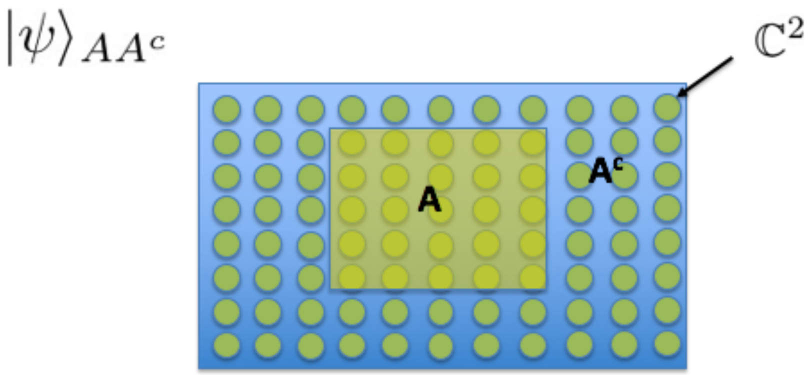
- Topological EE quantifies “non-local entanglement”

(Kitaev ‘12) $\gamma = 0$: state is adiabatically connected to trivial phase
(Kim ‘13) $\log(N) \leq 2\gamma$ N := number topologically protected states

- Bulk-boundary correspondence: topological order in the bulk has an effect on the boundary

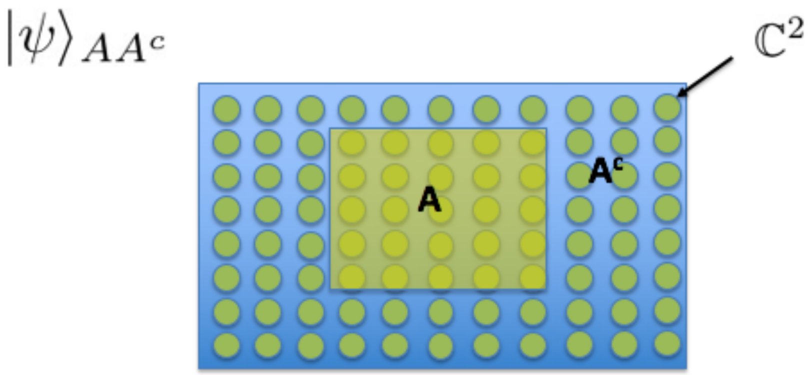
Area Law



Entanglement is "localized", concentrated around the boundary

What are the consequences of an area law?
What's the influence of TEE on the boundary?

Area Law



Entanglement is "localized", concentrated around the boundary

What are the consequences of an area law?
What's the influence of TEE on the boundary? This talk:

Area Law → TEE determines locality of
i) Boundary State
ii) Entanglement Spectrum

by strong subaddivitity and
stronger subaddivitity

Quantum Information 1.01: Fidelity

… it's a measure of distinguishability between two quantum states.

Given two quantum states their fidelity is given by

$$F(\rho,\sigma) := \mathrm{tr}((\rho^{1/2}\sigma\rho^{1/2})^{1/2})$$

It tells how distinguishable they are by any quantum Measurement

Ex 1: F=1: same state

Ex 2: F=0 : perfectly distinguishable states

Quantum Information 1.01: Relative Entropy

… it's another measure of distinguishability between two quantum states.

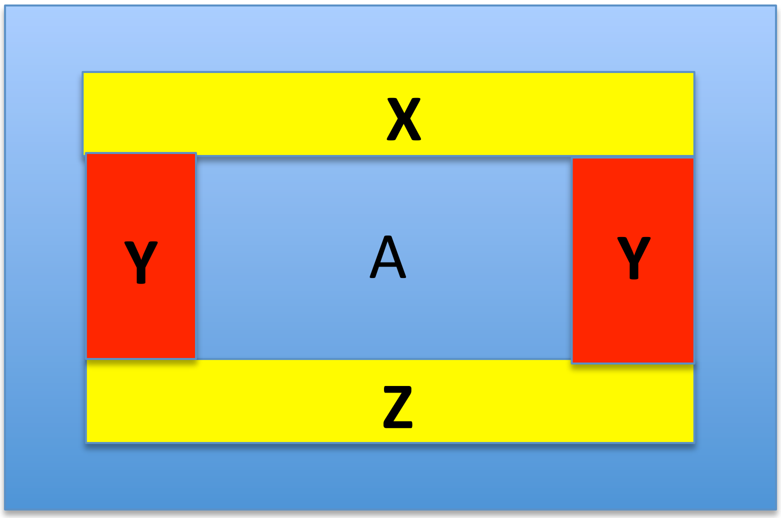
Def: $S(\rho\|\sigma) := \mathrm{tr}(\rho(\log(\rho) - \log(\sigma)))$

Gives optimal exponent for distinguishing the two states

Pinsker's inequality: $S(\rho\|\sigma) \geq -\frac{1}{2}\log F(\rho,\sigma)$

$$S(\rho\|\sigma) \approx 0 \implies \rho \approx \sigma$$

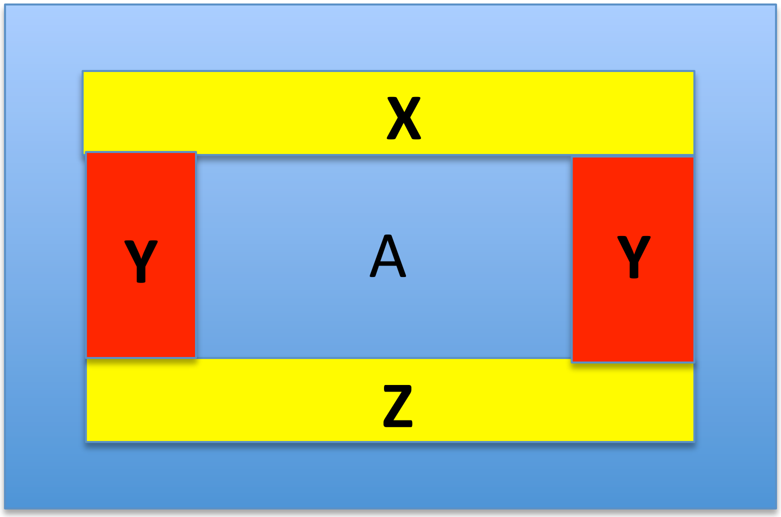
Topological EE and Locality of Boundary States



ρ_{XYZ}: reduced state on XYZ

XYZ Boundary of A

Topological EE and Locality of Boundary States



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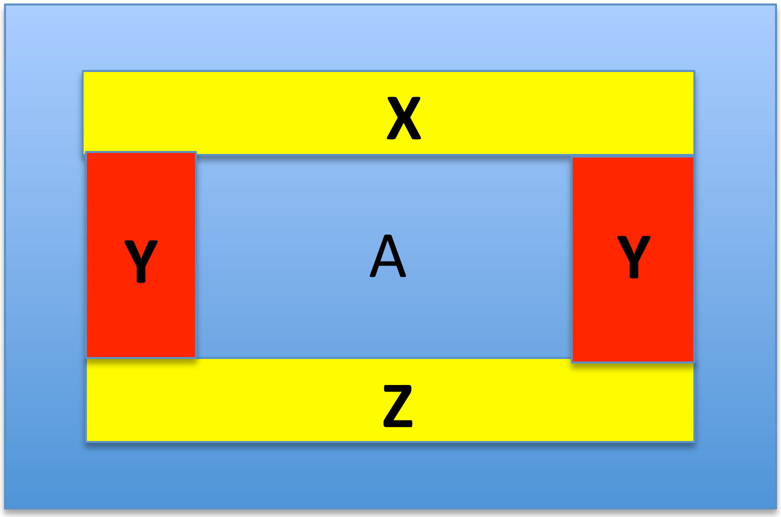
XYZ Boundary of A

Result 1. If $S(X) = \alpha|\partial X| - \gamma + \ldots$:

$$\gamma \approx \min_{H_{XY}, H_{YZ}} S(\rho_{XYZ} \| \exp(H_{XY} + H_{YZ})/\mathrm{tr}(\ldots))$$

Topological EE and Locality of Boundary States



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$e^{-|\partial X|/\xi}$

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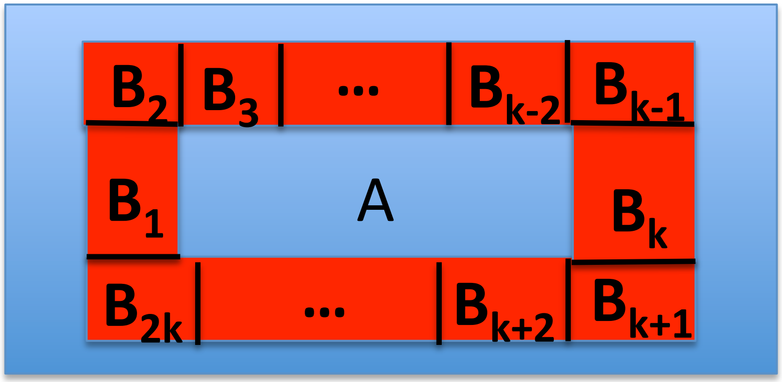
Topological EE and Locality of Boundary States



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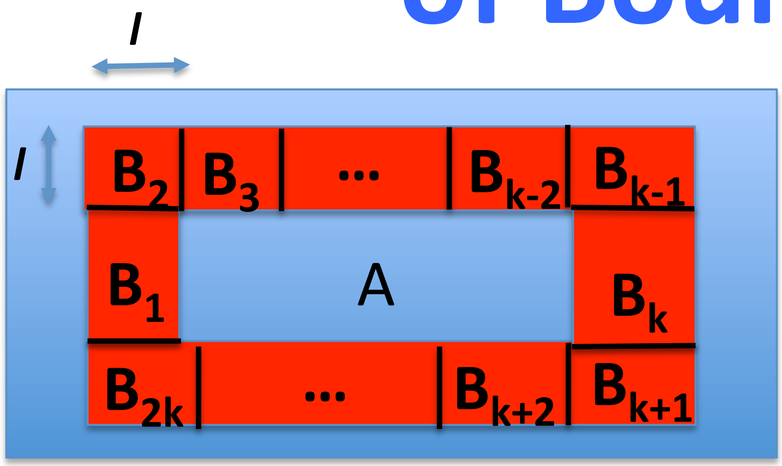
$$\approx \min_{H_{B_1 B_2}, \ldots, H_{B_{2k-1} B_{2k}}} S(\rho_{B_1 \ldots B_{2k}} \| \exp(H_{B_1 B_2} + \ldots + H_{B_{2k-1} B_{2k}})/\mathrm{tr}(\ldots))$$

Topological EE and Locality of Boundary States



ρ_{XYZ}: reduced state on XYZ

XYZ Boundary of A

$e^{-|\partial X|/\xi}$ $\quad l = O(\log(|A|))$

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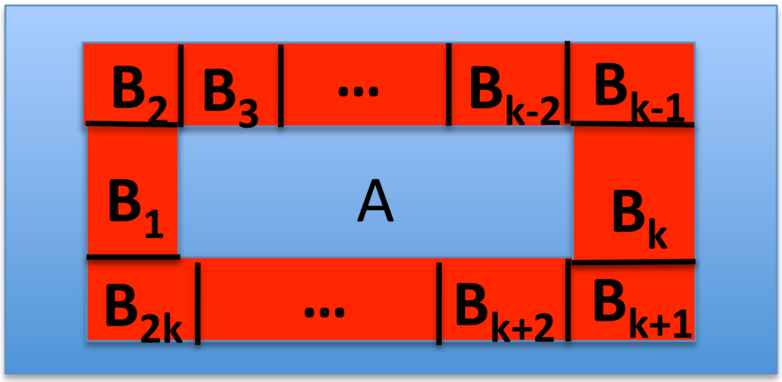
$e^{-|\partial X|/\xi}$

Topological EE and Locality of Boundary States



ρ_{XYZ}: reduced state on XYZ

XYZ Boundary of A

Obs 1: $\gamma = 0$

$$\Longrightarrow \quad \rho_B \approx \exp(H_{B_1 B_2} + \ldots H_{B_{2k-1} B_{2k}} / \mathrm{tr}(\ldots))$$

Obs 2: Thermal states has same on-site symmetries as original state

Obs 3: Thermal state is max entropy state consistent with local constraints

TEE gives number of non-local bits

Interpretation relative entropy (Anshu *et al* '14)

Alice

knows ρ

Bob

knows σ

TEE gives number of non-local bits

Interpretation relative entropy (Anshu *et al* '14)

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EPR pairs

Bob

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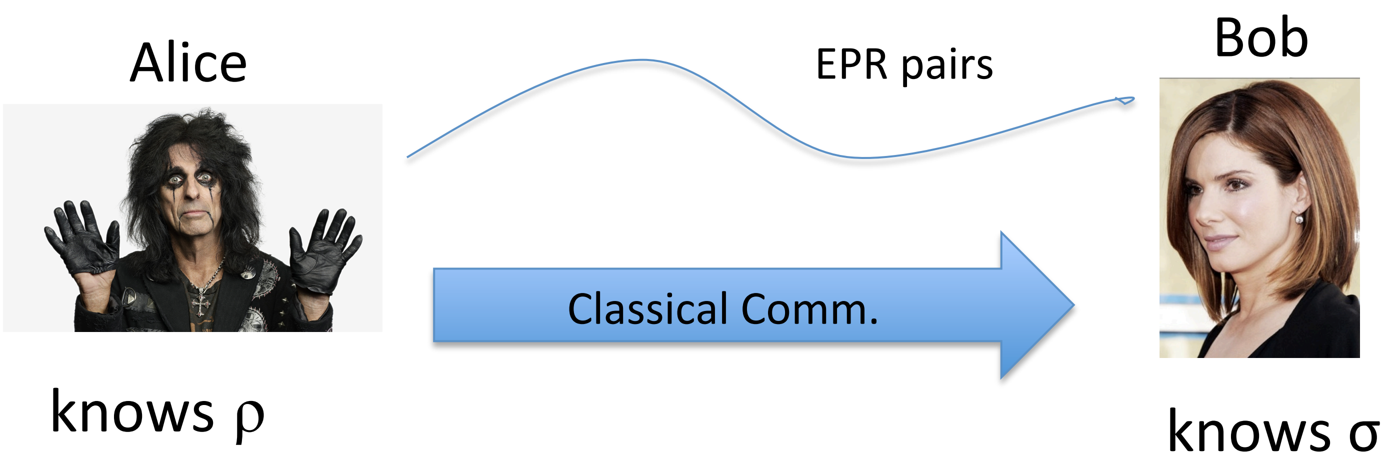
TEE gives number of non-local bits

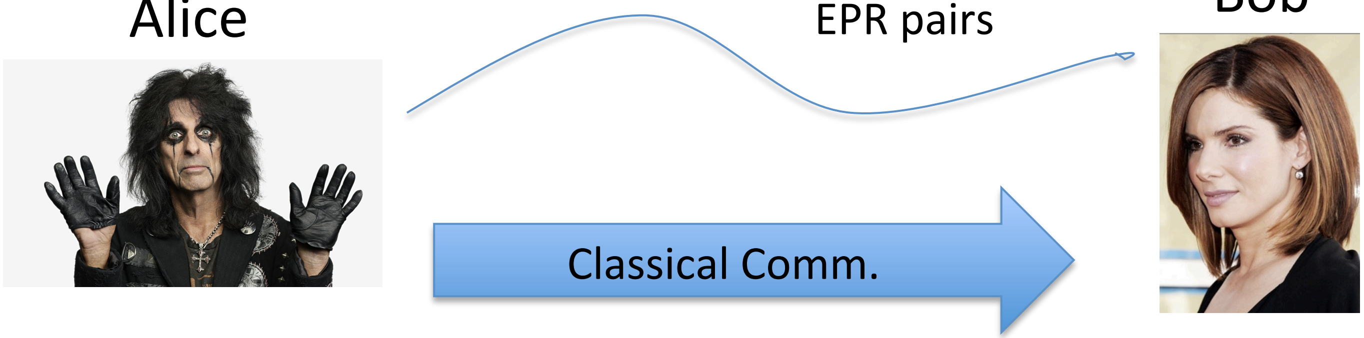
Interpretation relative entropy (Anshu *et al* ‘14)



TEE gives number of non-local bits

Interpretation relative entropy (Anshu *et al* ‘14)

Alice Bob



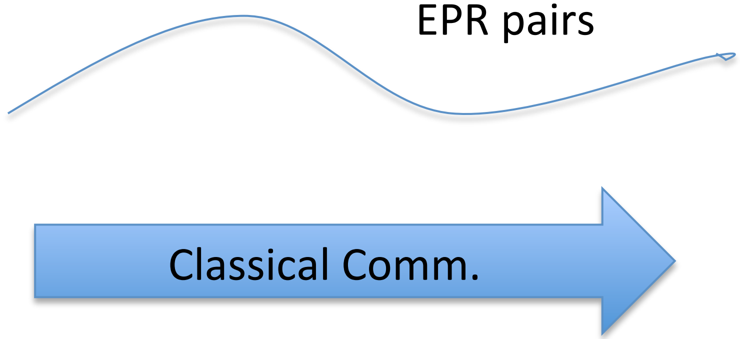
knows ρ knows σ

What’s the minimum classical comm. required for Bob to learn ρ?
(i.e. to be able to prepare a copy of ρ)

TEE gives number of non-local bits

Interpretation relative entropy (Anshu *et al* ‘14)

Alice



Bob

knows ρ

knows σ

$\approx S(\rho\|\sigma)$ necessary and sufficient for Bob to prepare a copy of ρ

TEE gives number of non-local bits

Interpretation relative entropy (Anshu *et al* '14)

Alice

EPR pairs

Classical Comm.

Bob

knows ρ

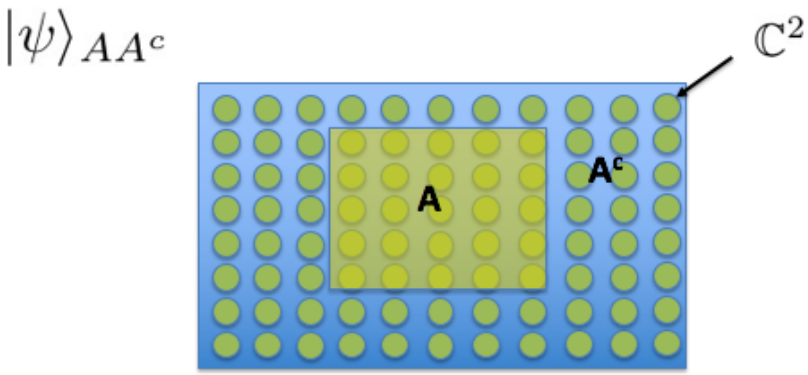
knows σ

$\approx S(\rho \| \sigma)$ necessary and sufficient for Bob to prepare a copy of ρ

$$\gamma \approx \min_{\sigma \in \text{Local Gibbs State}} S(\rho_{B_1 \ldots B_{2k}} \| \sigma)$$ gives number of non-local bits of ρ

obs: Consistent with γ = log(quantum dimension)

Entanglement Spectrum



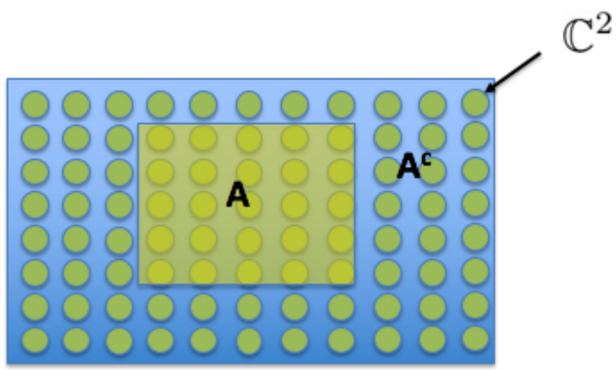
$\lambda(\rho_A)$: eigenvalues of ρ_A
Entanglement Spectrum

Area law statement about $-\sum_i \lambda_i \log \lambda_i$

What can we say about the whole spectrum?

Entanglement Spectrum

$|\psi\rangle_{AA^c}$



$\lambda(\rho_A)$: eigenvalues of ρ_A
Entanglement Spectrum

Area law statement about $-\sum_i \lambda_i \log \lambda_i$

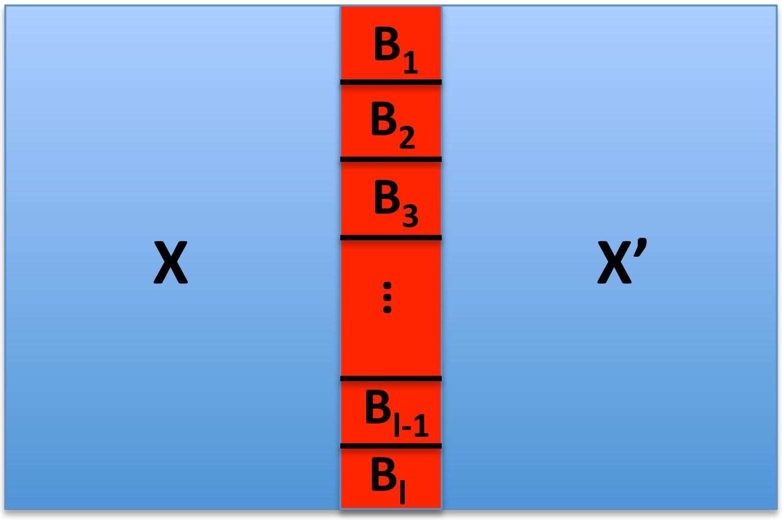
What can we say about the whole spectrum?

(Haldane, Li ’08, Cirac, Poiblanc, Schuch, Verstraete ’11, …)

γ=0: matches spectrum thermal state local model

γ≠0: matches spectrum thermal state local model
after projecting into topological superselection sector

Entanglement Spectrum



We assume translation invariance s.t. $\rho_X = \rho_{X'}$

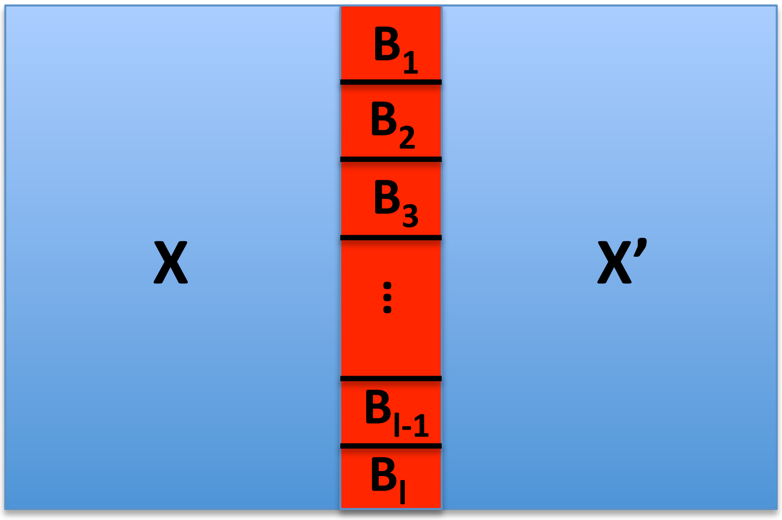
Result 2: If $S(X) = \alpha|\partial X| - \gamma + \ldots$:

$$\gamma = 0 \implies \lambda(\rho_X)^{\otimes 2} \approx \lambda(e^{\sum_k H_{B_k, B_{k+1}}})$$

$$\gamma \neq 0 \implies \lambda(\rho_X)^{\otimes 2} \approx \lambda(\sigma),$$

$$\mathrm{tr}_{B_1}(\sigma) = e^{\sum_{k>1} H_{B_k, B_{k+1}}}$$

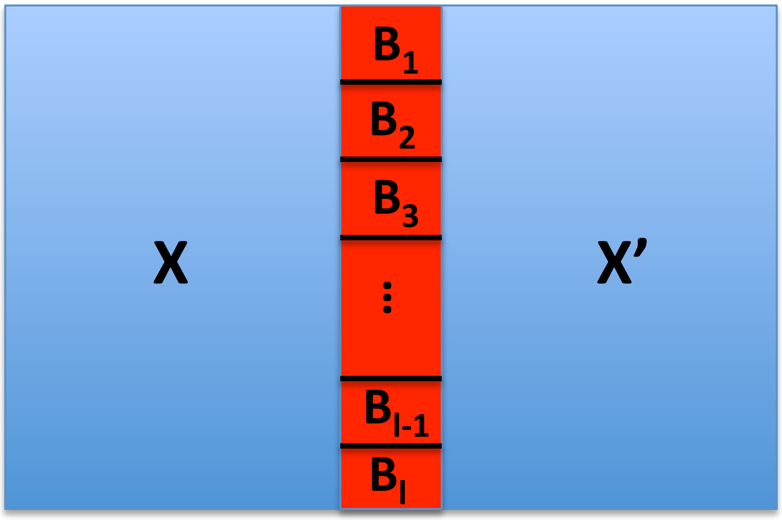
Result 2 from 1



From area law assumption:
(more later)

$$\rho_{XX'} \approx \rho_X \otimes \rho_{X'}$$

Result 2 from 1



From area law assumption:
(more later)

$$\rho_{XX'} \approx \rho_X \otimes \rho_{X'}$$

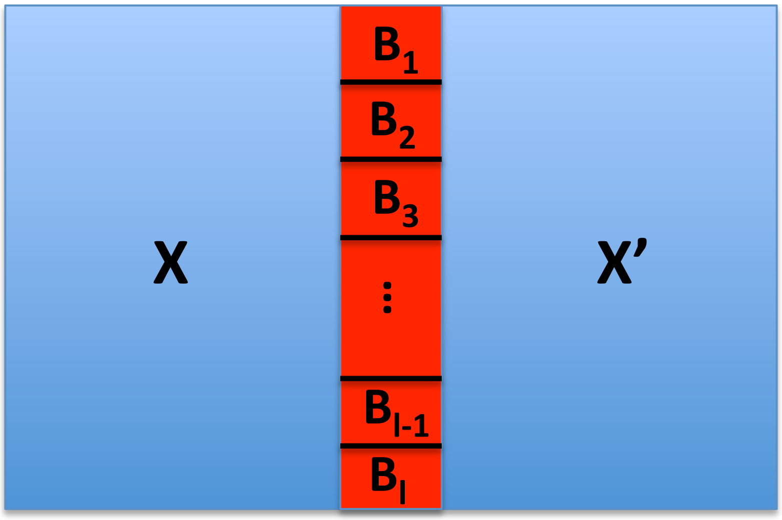
$$\lambda(\rho_{XX'}) = \lambda(\rho_B) \Longrightarrow \lambda(\rho_X) \otimes \lambda(\rho_{X'}) \approx \lambda(\rho_B)$$

Uhlmann's theorem There is an isometry U : B -> $B_X B_{X'}$ s.t.

$$U|\psi\rangle_{XBX'} \approx |\phi\rangle_{XB_X} \otimes |\phi'\rangle_{XB_{X'}} \quad \rho_X = \mathrm{tr}_{B_X}(|\phi\rangle\langle\phi|_{XB_X})$$

U maps degrees of freedom of X and X' into B

Result 2 from 1



From area law assumption:
(more later)

$$\rho_{XX'} \approx \rho_X \otimes \rho_{X'}$$

$$\lambda(\rho_{XX'}) = \lambda(\rho_B) \Rightarrow \lambda(\rho_X) \otimes \lambda(\rho_{X'}) \approx \lambda(\rho_B)$$

If $\gamma = 0$, $\rho_B \approx e^{\sum_k H_{B_k, B_{k+1}}} / Z$

$$\begin{aligned} \gamma &\approx \min_{H_{XY}, H_{YZ}} S(\rho_{XYZ} \| \exp(H_{XY} + H_{YZ})/\mathrm{tr}(\ldots)) \\ &\approx \min_{H_{B_1B_2}, \ldots, H_{B_{2k-1}B_{2k}}} S(\rho_{B_1 \ldots B_{2k}} \| \exp(H_{B_1B_2} + \ldots + H_{B_{2k-1}B_{2k}})/\mathrm{tr}(\ldots)) \end{aligned}$$

Why does it hold?

We want to show:

$$\begin{aligned} \gamma &\approx \min_{H_{XY}, H_{YZ}} S(\rho_{XYZ} \| \exp(H_{XY} + H_{YZ})/\mathrm{tr}(...)) \\ &\approx \min_{H_{B_1 B_2}, \ldots, H_{B_{2k-1} B_{2k}}} S(\rho_{B_1 \ldots B_{2k}} \| \exp(H_{B_1 B_2} + \ldots + H_{B_{2k-1} B_{2k}})/\mathrm{tr}(...)) \end{aligned}$$

ꭗ = 0 : follow from *strong subadditivity* (SSA) (Lieb, Ruskai '73)

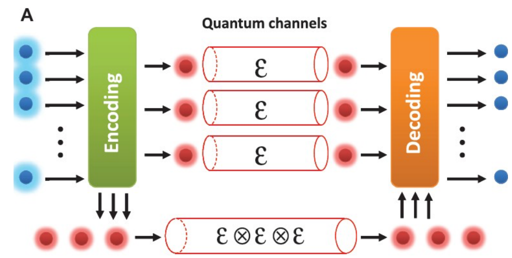
$$S(AB) + S(BC) \geq S(ABC) + S(B)$$

ꭗ ≠ 0 : follows from a *strengthening* of SSA (Fawzi and Renner '14)

Applications of SSA

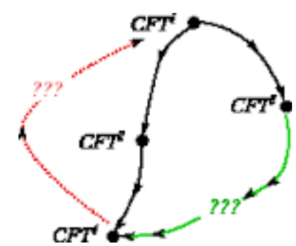
Used to prove optimal rates for nearly every quantum information protocol.

- Channel capacities (classical, quantum, private)
- Distillable Entanglement
- ....



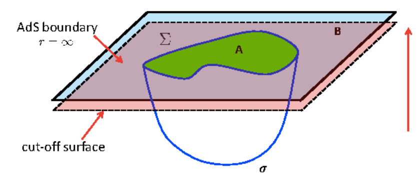
(Casini, Huerta, Myers ...) SSA + Lorentz Invariance:

- Entropic proof of the c-theorem (irreversibility of renormalization flow)



- Proof of Bekenstein's and Bousso's bound

(Ryu-Takayanagi, Headrick, ...) Test for holographic proposals of entropy



Many others...

Conditional Mutual Information

Given ρ_{ABC} ,

$$\begin{aligned} I(A:C|B) &:= S(AB) + S(BC) - S(ABC) - S(B) \\ &= S(\rho_{ABC} \| \exp(\log(\rho_{AB}) + \log(\rho_{BC}) - \log(\rho_B))) \end{aligned}$$

Strong subadditivity: $I(A:C|B) \geq 0$

Conditional Mutual Information

Given ρ_{ABC},

$$
\begin{aligned}
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\end{aligned}
$$

Strong subadditivity: $I(A:C|B) \geq 0$

Stronger subadditivity (Fawzi-Renner '14):

$$
I(A:C|B) \geq \frac{1}{2} \min_{\Lambda:B \to BC} -\log(F(\rho_{ABC}, \Lambda(\rho_{AB})))
$$

Conditional Mutual Information

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Stronger subadditivity (Fawzi-Renner '14):

$$I(A:C|B) \geq \frac{1}{2} \min_{\Lambda: B \to BC} -\log(F(\rho_{ABC}, \Lambda(\rho_{AB})))$$

$$I(A:C|B) \approx 0 \implies I_A \otimes \Lambda^{B \to BC}(\rho_{BC}) \approx \rho_{ABC}$$

quantum channel

Conditional Mutual Information

Given ρ_{ABC} ,

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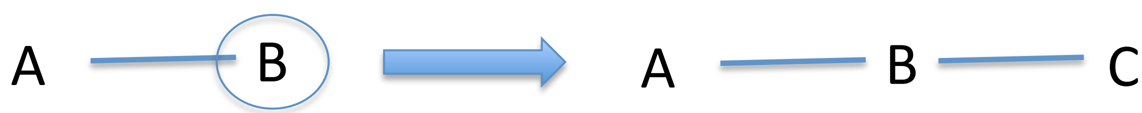
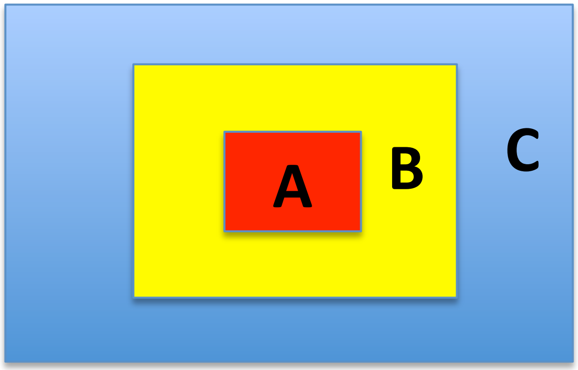
$$I(A:C|B) \geq \frac{1}{2} \min_{\Lambda:B \to BC} - \log(F(\rho_{ABC}, \Lambda(\rho_{AB})))$$

Can reconstruct the state ABC from reduction on AB by acting on B only

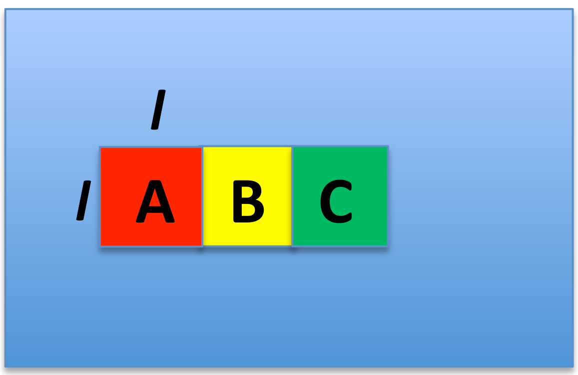


Consequence of Area Law: State Reconstruction



For every ABC with trivial topology:

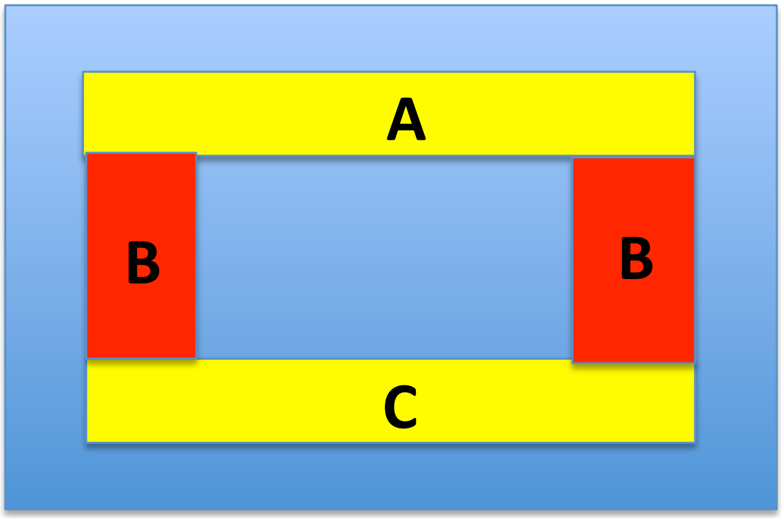
$$I(A:C|B) \approx 0$$



$$\begin{aligned} & I(A:C|B) \\ = \quad & S(AB) + S(BC) - S(ABC) - S(B) \\ = \quad & \alpha(|\partial(AB)| + |\partial(BC)||\partial(ABC)| - |\partial(B)|) + \ldots \\ = \quad & \alpha(6l + 6l - 8l - 4l) + \ldots \end{aligned}$$

TEE as Conditional Mutual Info

(Kitaev, Preskill '05, Levin, Wen '05)



$$\gamma = I(A:C|B) + \ldots$$

$$\begin{aligned} & I(A:C|B) \\ = \; & S(AB) + S(BC) - S(ABC) - S(B) \\ = \; & \alpha(\partial(AB) + |\partial(BC)| - |\partial(ABC)| - |\partial(B)|) - \gamma - \gamma + \gamma + 2\gamma + \ldots \\ = \; & \gamma + \ldots \end{aligned}$$

Non zero TEE gives an obstruction to reconstruct ρ_{ABC} from ρ_{AB} by acting on B

Why does it work?

We want to show:

$$
\begin{aligned}
\gamma \quad &\approx \min_{H_{XY}, H_{YZ}} S(\rho_{XYZ} \| \exp(H_{XY} + H_{YZ})/\mathrm{tr}(\ldots)) \\
&\approx \min_{H_{B_1 B_2}, \ldots, H_{B_{2k-1} B_{2k}}} S(\rho_{B_1 \ldots B_{2k}} \| \exp(H_{B_1 B_2} + \ldots + H_{B_{2k-1} B_{2k}})/\mathrm{tr}(\ldots))
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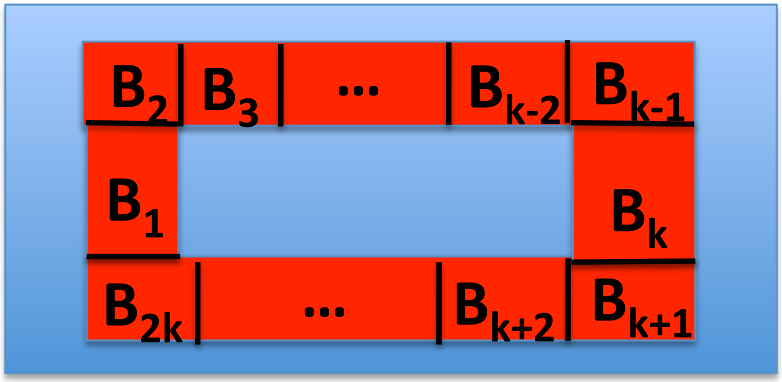
$$\begin{aligned} \gamma &\approx \min_{H_{XY}, H_{YZ}} S(\rho_{XYZ} \| \exp(H_{XY} + H_{YZ})/\mathrm{tr}(\ldots)) \\ &\approx \min_{H_{B_1 B_2}, \ldots, H_{B_{2k-1} B_{2k}}} S(\rho_{B_1 \ldots B_{2k}} \| \exp(H_{B_1 B_2} + \ldots + H_{B_{2k-1} B_{2k}})/\mathrm{tr}(\ldots)) \end{aligned}$$

Let's start with the case ɣ=0.



Need to show $\rho_{B_1 \ldots B_{2k}}$ is close to thermal assuming all conditional mutual information are small, i.e. **approximately independence**

$$I(B_1 \ldots B_{j-1} : B_{j+1} \ldots B_{2k-1} | B_j B_{2k}) \approx 0$$

Markov Chain

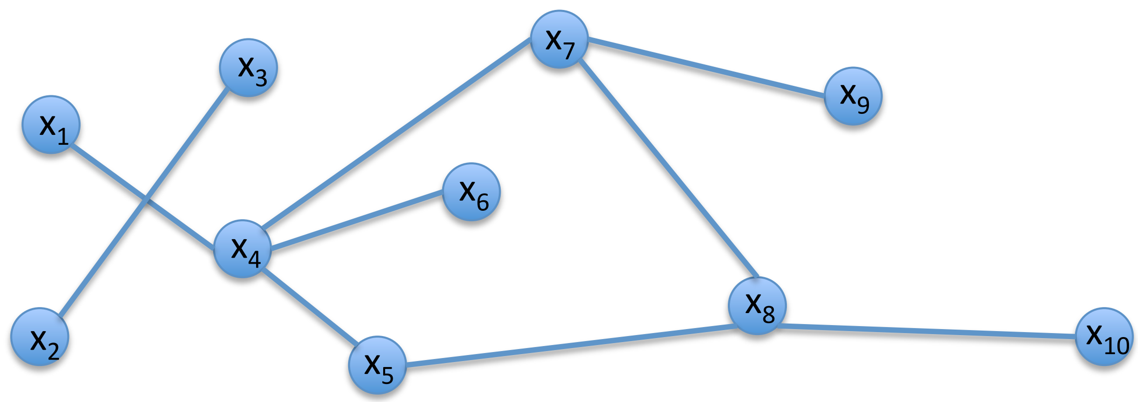
X, Y, Z with distribution p(x, y, z)

i) X-Y-Z Markov if X and Z are independent conditioned on Y

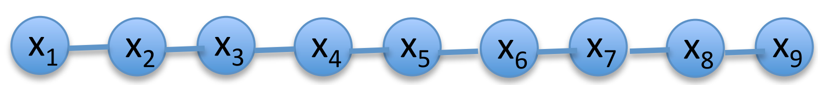
ii) X-Y-Z Markov if there is a channel $\Lambda : Y \to YZ$ s.t. $\Lambda(p_{XY}) = p_{XYZ}$

iii) $I(X:Y|Z)_p = \mathbb{E}_{z\sim p(z)} I(X:Y)_{p(x,y|z=z')}$

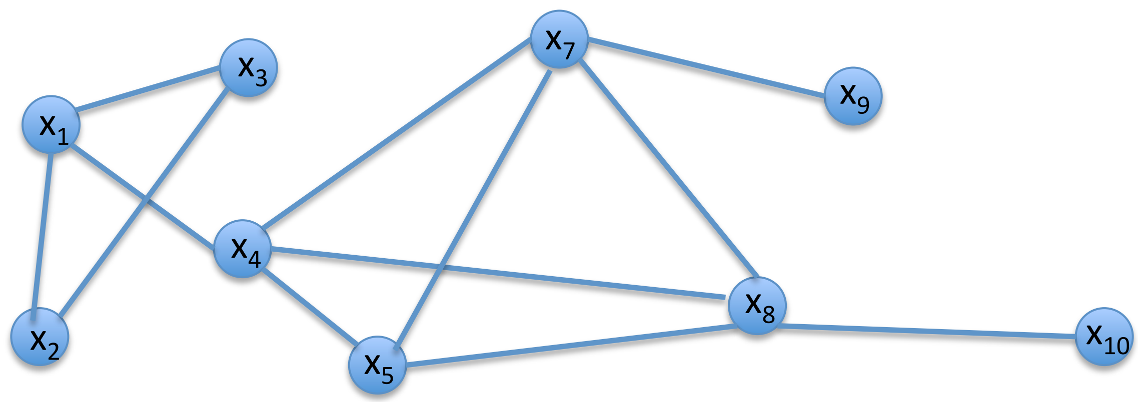
Markov Networks



We say $X_1, \ldots, X_n$ on a graph G form a Markov Network if X_i *is* indendent of all other X's conditioned on its neighbors

Ex: Markov chains 

Hammersley-Clifford Theorem

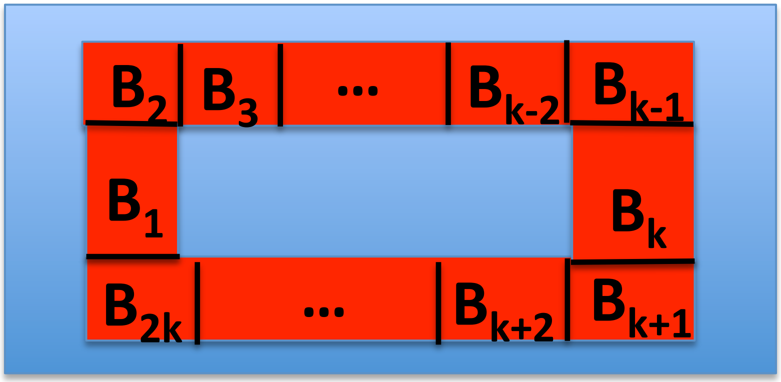


Markov networks $\Longleftrightarrow$ Gibbs state local classical Hamiltonian

(on cliques of the graph)

Going Back

Need to show $\rho_{B_1 \ldots B_{2k}}$
is close to thermal assuming all
conditional mutual information
are small (approximately independence)



$$I(B_1 \ldots B_{j-1} : B_{j+1} \ldots B_{2k-1} | B_j B_{2k}) \approx 0$$

We want a **quantum** and **approximate** version of **Hammersley-Clifford**,
but only for 1D chains

Quantum Markov Chain

Classical: X, Y, Z with distribution p(x, y, z)

i) X-Y-Z Markov if X and Z are independent conditioned on Y

ii) X-Y-Z Markov if there is a channel $\Lambda : Y \to YZ$ s.t. $\Lambda(p_{XY}) = p_{XYZ}$

Quantum: (Hayden, Jozsa, Petz, Winter '03)

i) ρ_{ABC} Markov quantum state if A and C are "independent conditioned" on B

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i) ρ_{ABC} Markov quantum state if A and C are "independent conditioned" on B, i.e. $H_B \simeq \bigoplus_k H_{B_{L,k}} \otimes H_{B_{R,k}}$ and

$$\rho_{ABC} = \bigoplus_k p_k \rho_{AB_{L,k}} \otimes \rho_{B_{R,k}C}$$

Quantum Markov Chain

Classical: X, Y, Z with distribution p(x, y, z)

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Quantum Markov Chain

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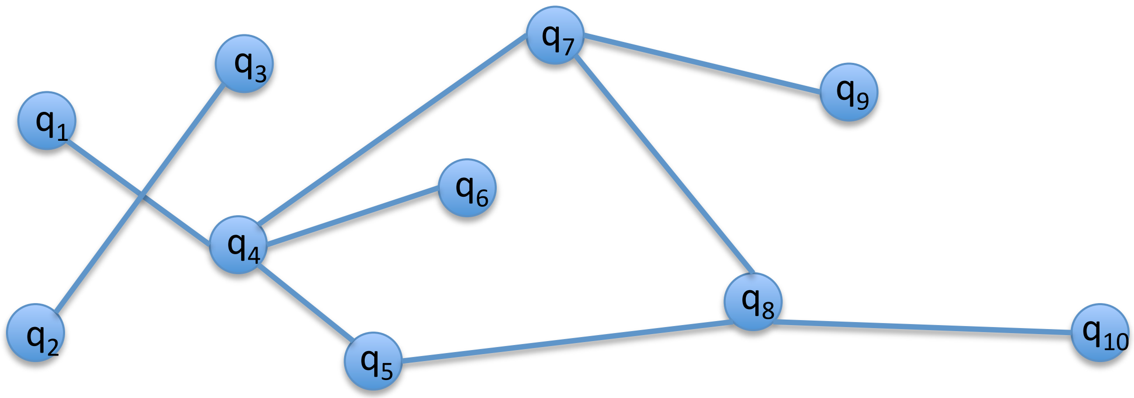
ii) ρ_{ABC} Markov if there is channel Λ : B -> BC s.t. $\Lambda(\rho_{AB}) = \rho_{ABC}$

iii) ρ_{ABC} Markov if $\rho_{ABC} = e^{H_{AB}+H_{BC}}, \quad [H_{AB}, H_{BC}] = 0$

Quantum Hammersley-Clifford Theorem

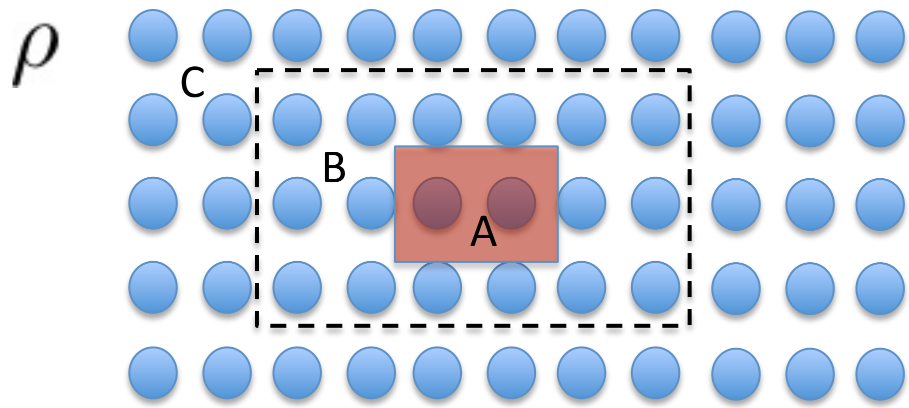


(Leifer, Poulin '08, Brown, Poulin '12) Analogous result holds replacing classical Hamiltonians by *commuting* quantum Hamiltonians

(obs: quantum version more fragile; only works for graphs with no 3-cliques)

Only Gibbs states of commuting Hamiltonians appear. Is there a fully quantum formulation?

Q. Approximate Markov States



ρ quantum approximate Markov if for every A, B, C

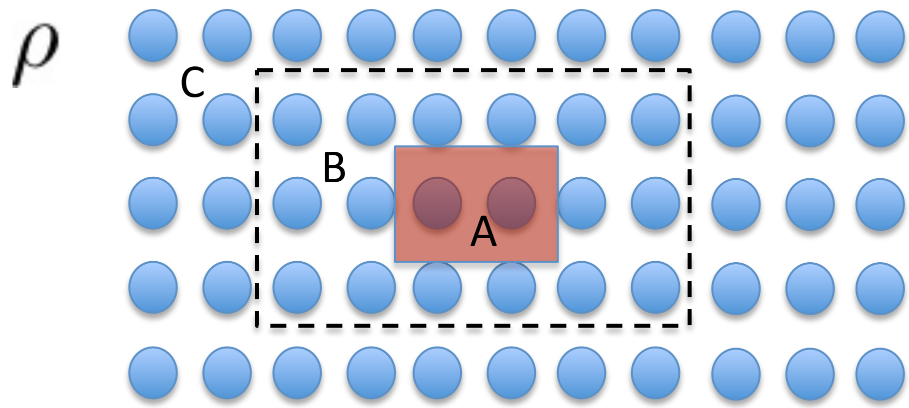
$$I(A:C|B) \to 0 \text{ when } \mathrm{dist}(A,C) \to \infty$$

Conjecture

Quantum Approximate Markov ⟺ Gibbs state local Hamiltonian

$$\rho = e^{\sum_k H_k}$$

Strengthening of Area Law



Conjecture

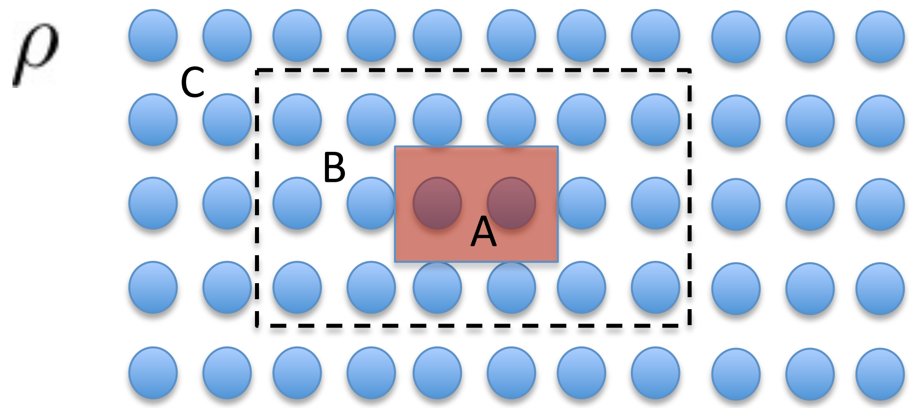
Quantum Approximate Markov ⟸ Gibbs state local Hamiltonian

(Wolf, Verstraete, Hastings, Cirac '07) $I(A:BC)_{\rho_T} \leq \frac{c}{T}|\partial A|$

Gibbs state @ temperature *T*: $$\rho_T := e^{-H/T}/Z$$

$$H = \sum_k H_k, \quad \|H_k\| \leq 1$$

Strengthening of Area Law



Conjecture

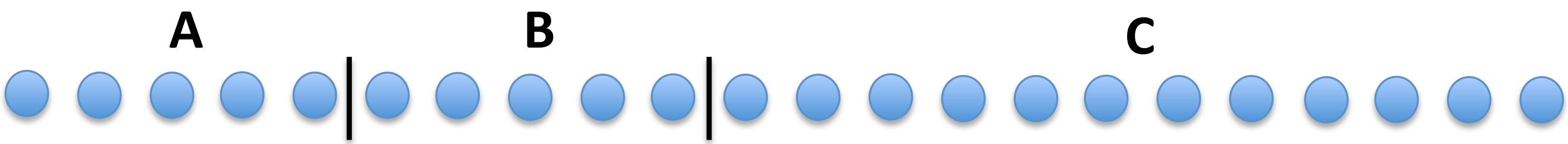
Quantum Approximate Markov ⟵ Gibbs state local Hamiltonian

From conjecture:

$$I(A:BC) = I(A:B) + I(A:C|B) \approx I(A:B)$$

Gives rate of saturation of area law

Approximate Quantum Markov Chains are Thermal

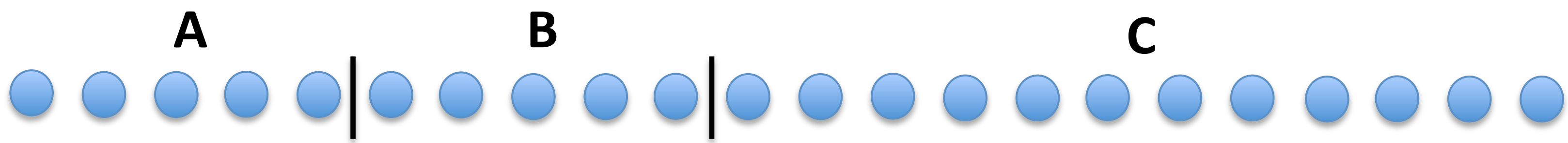


thm

1. Let *H* be a local Hamiltonian on *n* qubits. Then

$$I(A:C|B)_{\rho_T} \leq e^{-c'\sqrt{|B|}+e^{c/T}}$$

Approximate Quantum Markov Chains are Thermal



thm

1. Let *H* be a local Hamiltonian on *n* qubits. Then

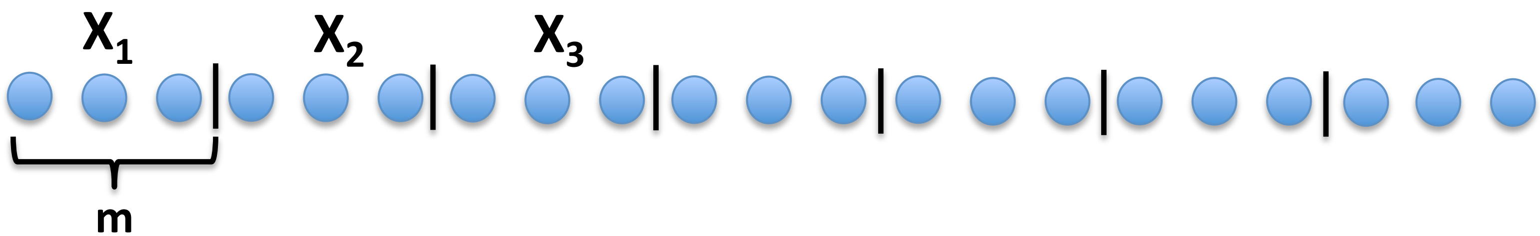
$$I(A:C|B)_{\rho_T} \leq e^{-c'\sqrt{|B|}+e^{c/T}}$$

2. Let $\rho_{1\ldots n}$ be a state on *n* qubits s.t. for every split ABC with |B| $I(A:C|B) \leq \varepsilon$. Then

$$\min_{H \in \mathcal{H}_{2m}} S\left(\rho || e^{H}\right) \leq \varepsilon \frac{n}{m}$$

$$\mathcal{H}_{2m} := \{H : H = \sum_k H_{k,k+1}, \ \forall k \ \mathrm{supp}(H_{k,k+1}) \leq 2m\}$$

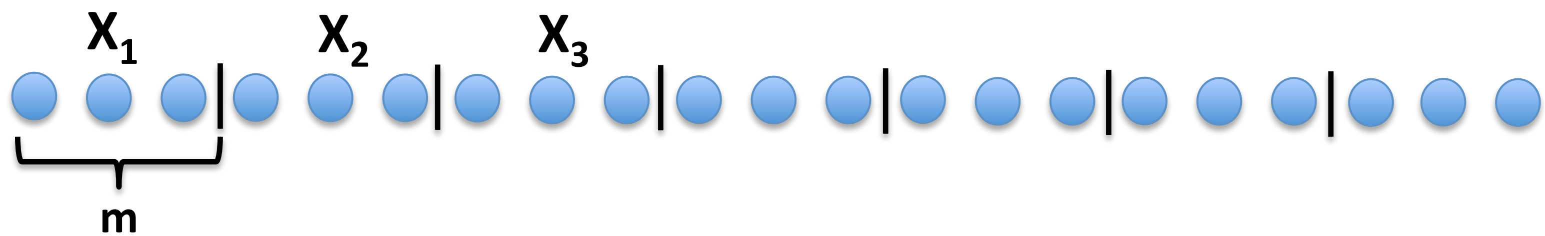
Proof Part 2



Let $\sigma_{X_1 \ldots X_{\frac{n}{m}}}$ be the maximum entropy state s.t.

$$\sigma_{X_i, X_{i+1}} = \rho_{X_i, X_{i+1}} \quad \forall i \in [n/m]$$

Proof Part 2



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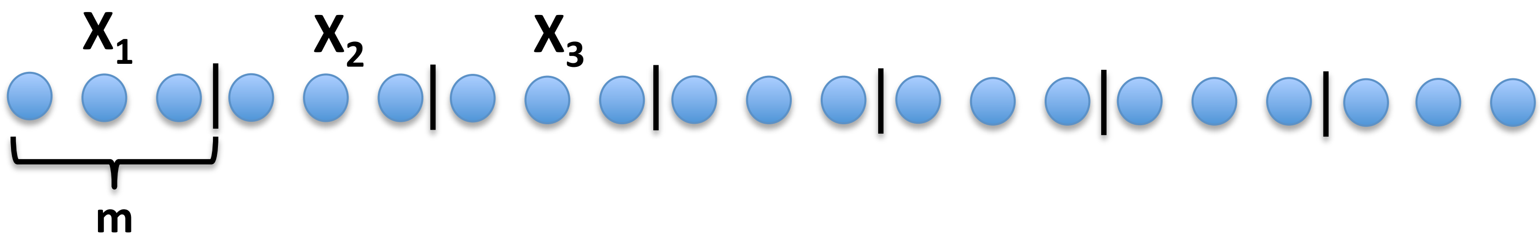
$$\sigma_{X_i, X_{i+1}} = \rho_{X_i, X_{i+1}} \quad \forall i \in [n/m]$$

Fact 1 (Jaynes '57): $\sigma = e^{\sum_k H_{X_k, X_{k+1}}}$

"maximum entropy state given linear constraints is thermal"

$$\operatorname{argmax}\left(S(\sigma) \text{ s.t. } \operatorname{tr}(\sigma M_i) = c_i\right) = \exp\left(\sum_i \lambda_i M_i\right)$$

Proof Part 2



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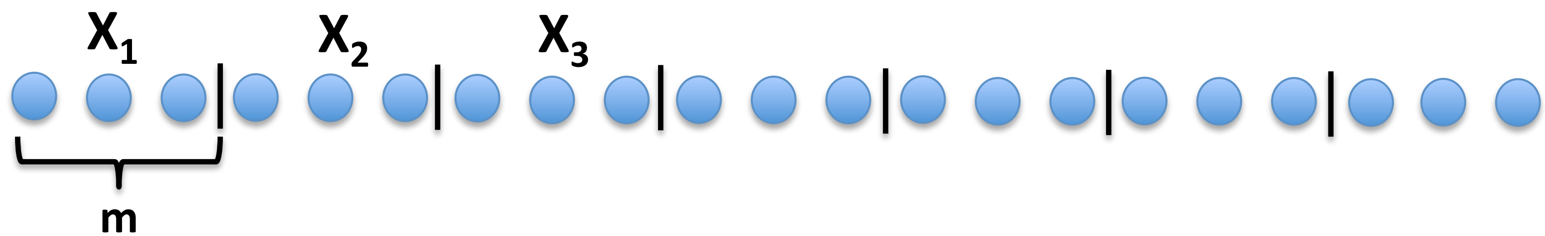
Fact 1 (Jaynes '57): $\sigma = e^{\sum_k H_{X_k, X_{k+1}}}$

Fact 2
$$\min_{H \in \mathcal{H}_{2m}} S(\rho \| e^H / Z) \leq -S(\rho) - \mathrm{tr}(\rho \log \sigma)$$
$$= S(\sigma) - S(\rho)$$

Let's show it's small

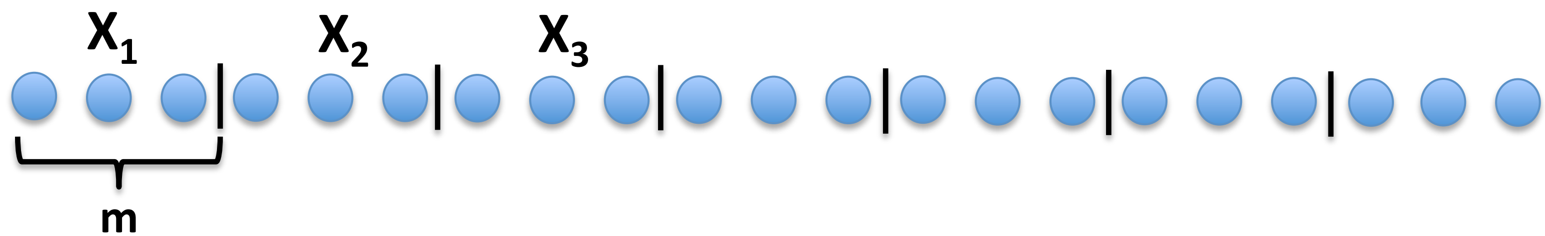
Proof Part 2



$$
\begin{aligned}
& S(X_1 \ldots X_{n/m})_\sigma \\
\leq \quad & S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 \ldots X_{n/m})_\sigma
\end{aligned}
$$

SSA

Proof Part 2



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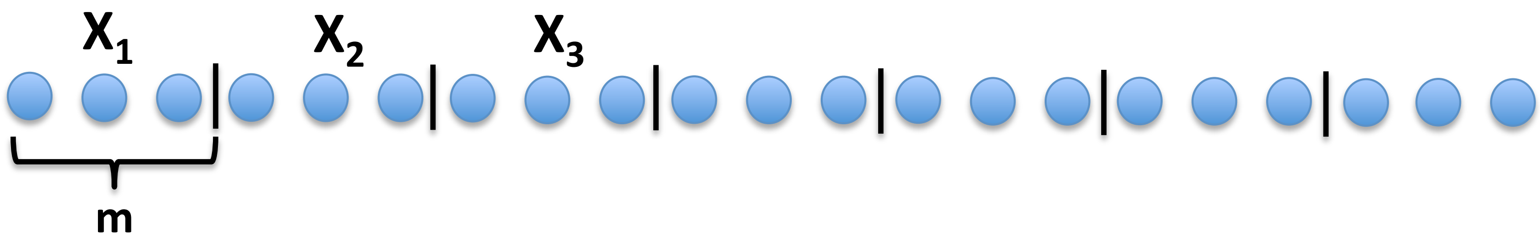
\end{aligned}
$$

Proof Part 2



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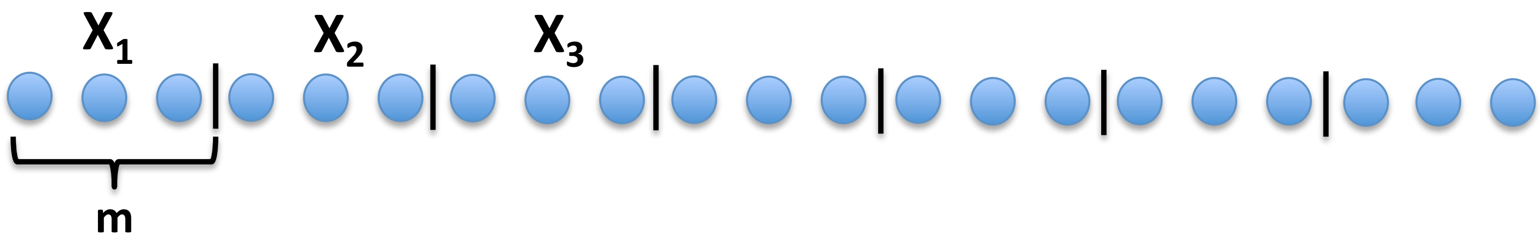
\leq\ & \sum_i S(X_i X_{i+1})_\sigma - S(X_{i+1})_\sigma
\end{aligned}
$$

Proof Part 2



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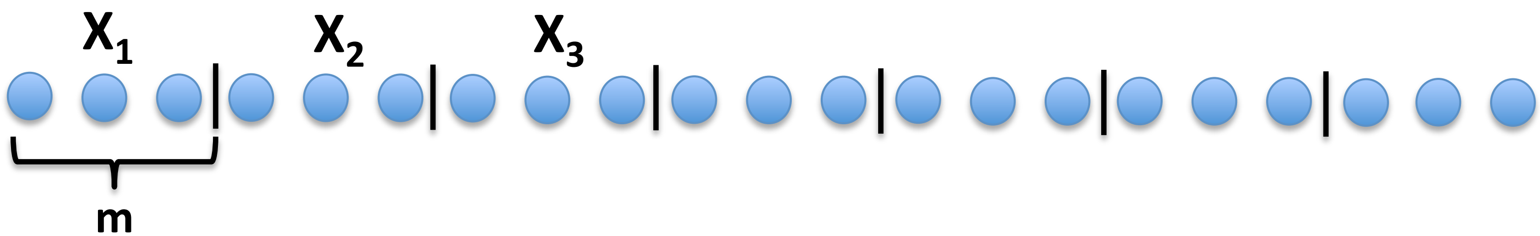
= \quad & \sum_i S(X_i X_{i+1})_\rho - S(X_{i+1})_\rho
\end{aligned}
$$

Since $\sigma_{X_i, X_{i+1}} = \rho_{X_i, X_{i+1}} \quad \forall i \in [n/m]$

Proof Part 2



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=\ & \sum_i S(X_i X_{i+1})_\rho - S(X_{i+1})_\rho \\
\leq\ & S(X_1 \ldots X_{n/m})_\rho + \varepsilon \frac{n}{m}
\end{aligned}
$$

Since $I(X_i : X_{i+2} \ldots X_{n/m} | X_{i+1}) \leq \varepsilon \ \forall i$

Proof Part 1

Recap: Let *H* be a local Hamiltonian on *n* qubits. Then

$$I(A:C|B)_{\rho_T} \leq e^{-c'\sqrt{|B|}+e^{c/T}}$$

We show there is a recovery channel from B to BC reconstructing the state on ABC from its reduction on AB.

More technical. Uses **Quantum Belief Propagation** equations of Hastings.

Summary

- Locality of EE (area law) implies locality of boundary states and entanglement spectrum
- Quantum Approximate Markov Chains are Thermal

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- Are two copies of entanglement spectrum needed?
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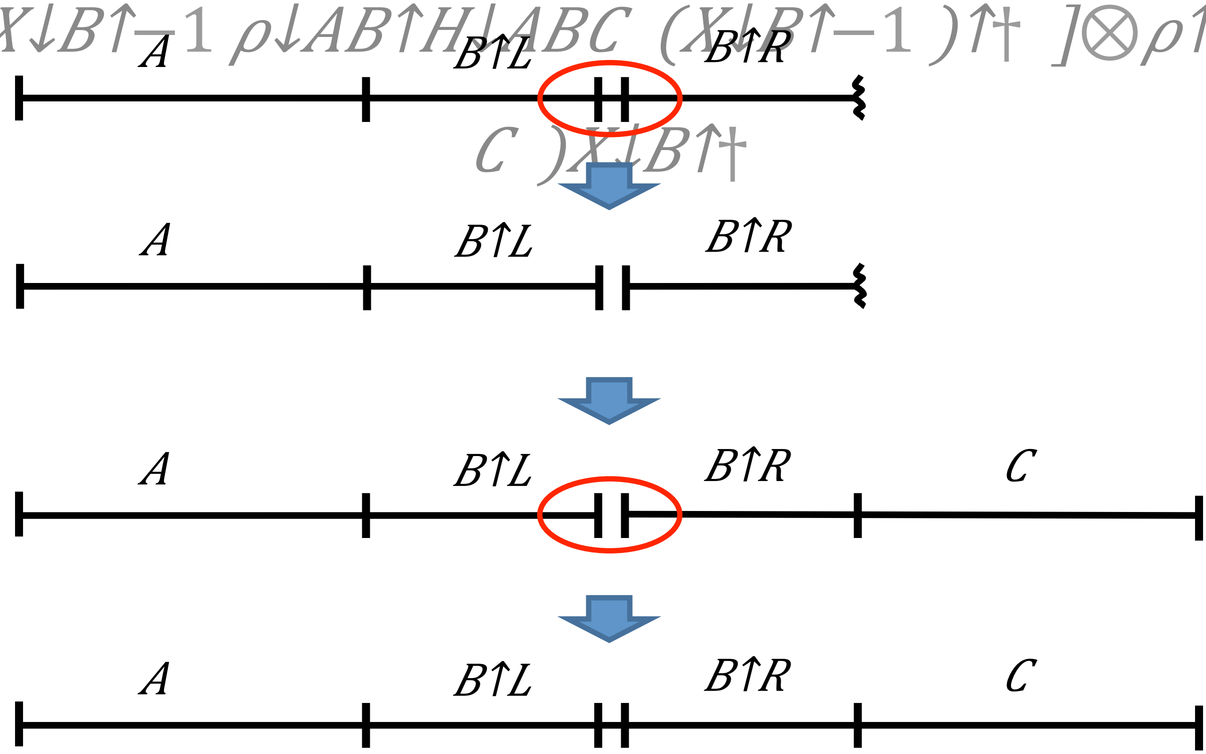
- Thermal state has same symmetries as original state. Mapping from 2D (zero temperature) to 1D (thermal). Is it useful for classification of (symmetry-protected) phases?

Structure of Recovery Map

There exists an operator X_B such that

$$\rho^{H_{ABC}} \approx \mathrm{id}_A \otimes \kappa_{B\to \mathrm{BC}}\left(\rho_{AB}^{H_{ABC}}\right) = X_B\left(\mathrm{tr}_{B^R}\left[X_B^{-1}\,\rho_{AB}^{H_{ABC}}\left(X_B^{-1}\right)^{\dagger}\right]\otimes \rho^{H_{B^R C}}\right)X_B^{\dagger}$$



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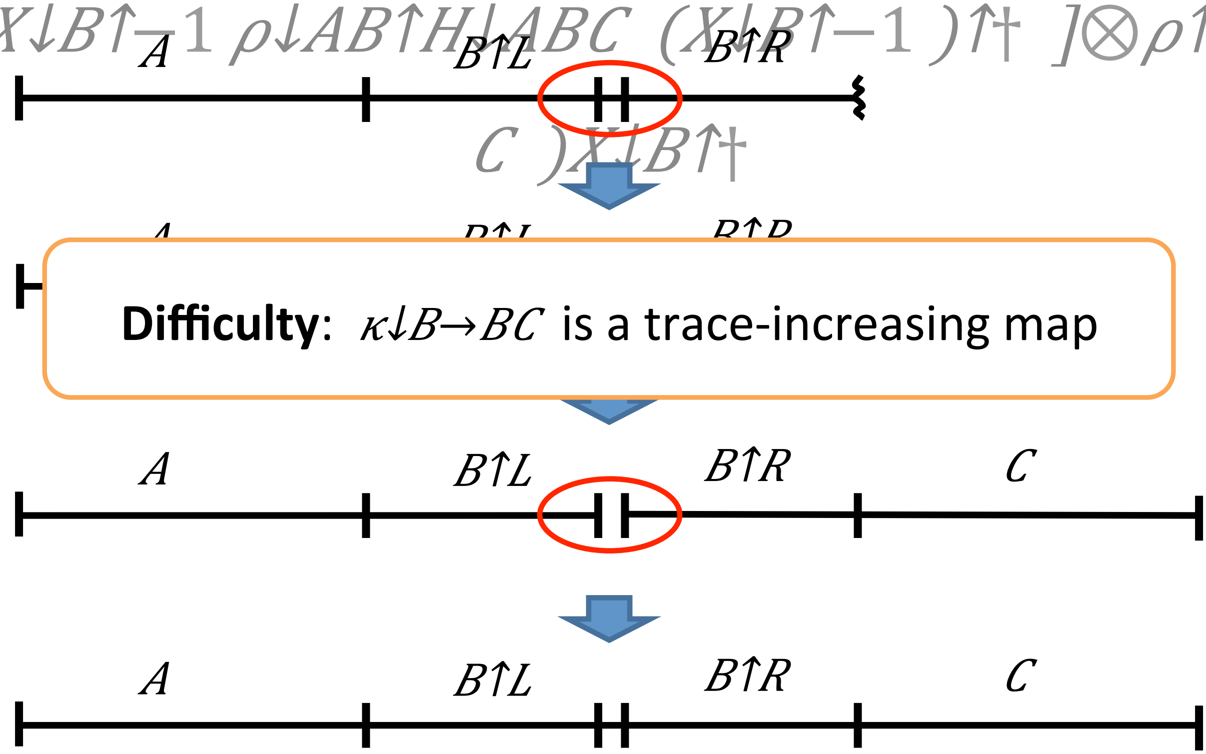
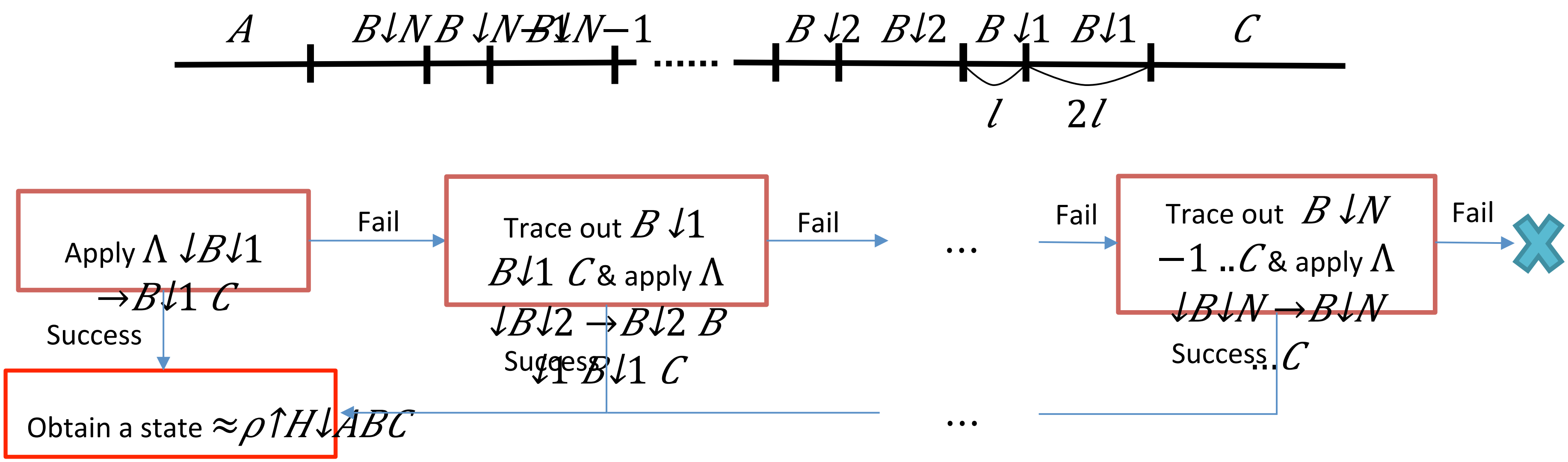


Difficulty: $\kappa_{B\to BC}$ is a trace-increasing map

Repeat-until-success Method

We normalize $\kappa_{B\to BC}$ and define a CPTD-map $\Lambda_{B\to BC}$.
→ Succeed to recover with a constant probability p.



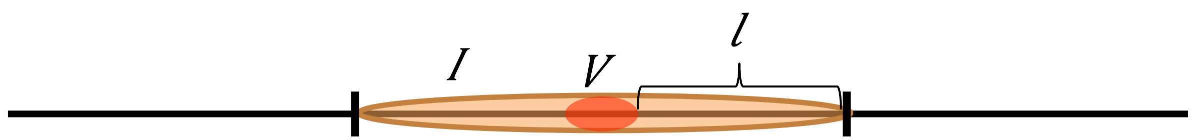
❑ Choose $N \sim l$ $(|B| = O(l^2))$.

→Total error=Fail probability $(1-p)^l$ + approx. error $O(e^{-O(l)}) = O(e^{-O(l)})$.

Locality of Perturbations

The key point in the proof:

For a short-ranged Hamiltonian H, the local perturbation to H only perturb the Gibbs state locally.



A useful lemma by Araki (Araki, '69)
For 1D Hamiltonian with short-range interaction H,

$$\| e^{H+V} e^{-H} - e^{H_I+V} e^{-H_I} \| \leq \mathcal{O}(e^{-\mathcal{O}(l)})$$

$$e^{-\beta H} \to e^{-\beta(H+V)} \approx X_I e^{-\beta H} X_I^{\dagger}$$

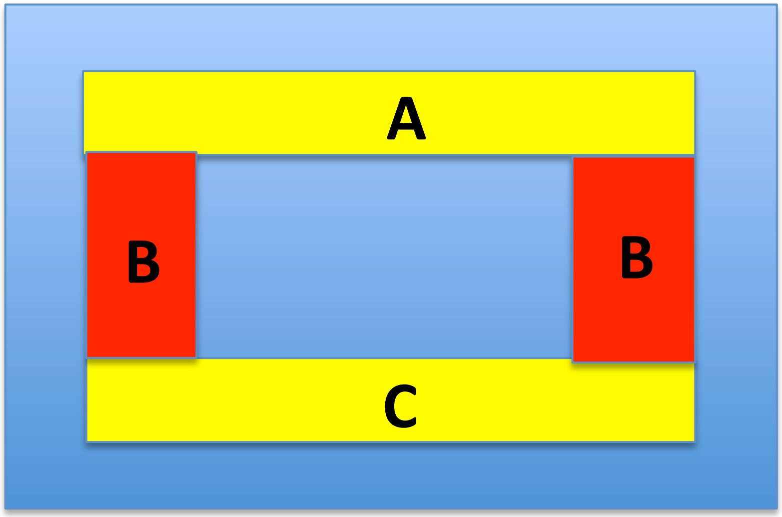
$$X_I = e^{-\beta/2\,(H_I+V)} e^{\beta/2\, H_I}$$

← Local

Proof for γ≠0

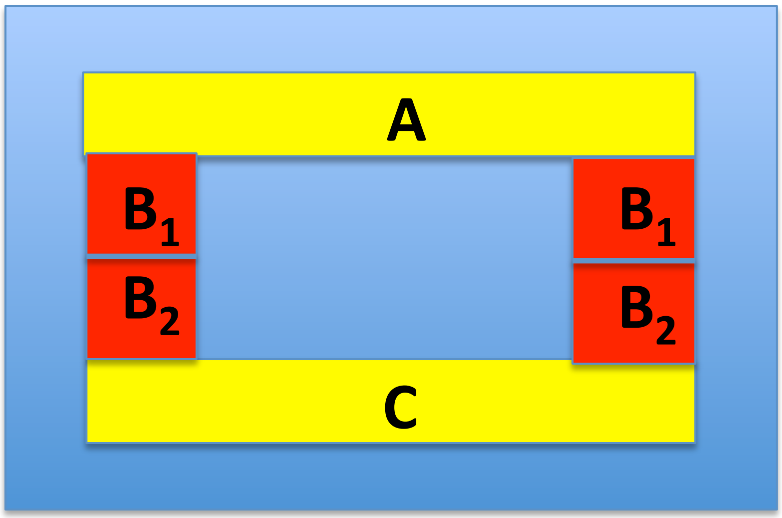
thm 1 Suppose $|\psi\rangle$ satisfies the area law assumption. Then

$$\begin{aligned} 2\gamma &\approx I(A:C|B) \\ &\approx \min_{H_{AB},H_{BC}} S(\rho_{ABC} \| \exp(H_{AB}+H_{BC})/Z) \end{aligned}$$



Proof for γ≠0

We follow the strategy of (Kato et al ‘15) for the zero-correlation length case



Area Law implies

$$I(A : B_2|B_1) \approx 0$$

$$I(C : B_1|B_2) \approx 0$$

By Fawzi-Renner Bound, there are channels $\Lambda : B_1 \to B_1 A$, $\Delta : B_2 \to B_2 C$ s.t.

$$\Lambda(\rho_{B_1 B_2}) \approx \rho_{AB_1B_2}, \quad \Delta(\rho_{B_1B_2}) \approx \rho_{B_1B_2C}$$

Proof for γ≠0

Define: $\sigma_{AB_1B_2C} := \Lambda^{B_1\to B_1A} \otimes \Delta^{B_2\to B_2C}(\rho_{B_1B_2})$

We have $\rho_{AB} \approx \sigma_{AB},\ \rho_{BC} \approx \sigma_{BC}$

It follows that C can be reconstructed from B. Therefore

$$I(A:C|B)_\sigma \approx 0$$

Proof for γ≠0

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It follows that C can be reconstructed from B. Therefore

$$I(A:C|B)_\sigma \approx 0$$

Since

$$I(A:C|B)_\sigma = S(\sigma_{ABC} \| \exp(\log(\sigma_{AB})) + \log(\sigma_{BC})) - \log(\sigma_B)))$$

$\pi \approx \sigma$ with

$$\pi := \exp(\log(\sigma_{AB}) + \log(\sigma_{BC}) - \log(\sigma_B))/\mathrm{tr}(\ldots)$$

So $I(A:C|B)_\pi \approx 0$

Proof for ɣ≠0

Since $I(A:C|B)_\pi \approx 0$

$$\begin{aligned} S(ABC)_\pi &\approx S(AB)_\pi + S(BC)_\pi - S(B)_\pi \\ &\approx S(AB)_\rho + S(BC)_\rho - S(B)_\rho \\ &= S(ABC)_\rho + I(A:C|B)_\rho \end{aligned}$$

Let R_2 be the set of Gibbs states of Hamiltonians $H = H_{AB} + H_{BC}$. Then

$$\begin{aligned} \min_{\nu \in R_2} S(\rho \| \nu) &= \min_{\nu \in R_2} -S(\rho) - \mathrm{tr}(\rho \log \nu) \\ &\approx I(A:C|B)_\rho + \min_{\nu \in R_2} -S(\pi) - \mathrm{tr}(\rho \log \nu) \\ &\approx I(A:C|B)_\rho + \min_{\nu \in R_2} -S(\pi) - \mathrm{tr}(\pi \log \nu) \\ &= I(A:C|B)_\rho \end{aligned}$$

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